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On a New Method of Quantisation and Some of its Applications

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Abstract

Any quantum-mechanical problem with $O(2,1)$ as SGA (spectrum-generating algebra) is considered as a single oscillator related to a new quantisation. In the case of small interactions the problems can be solved within essentially Fock representations while in the case of strong attractive potentials they can be solved only within the essentially non-Fock representations of the new commutation relations. Explicit realisations of a system of n oscillators through para-Bose operators have been constructed.

1. *Introduction*

In 1953 Green (Green, 1953) introduced the para-Fermi and para-Bose quantisation which generalise the Fermi and Bose quantisation respectively. It has been proved (Ryan & Sudarshan, 1963) that the algebra of the para-Fermi quantisation with $2n$ generators is isomorphic to $O(2n+1)$ algebra.

A new method of quantisation has been recently proposed (Kademova & Kraev, 1971a, b) consisting of the following commutation relations:

$$[\tfrac{1}{2}[\mathscr{F}_i, \overset{+}{\mathscr{F}}_j]_-, \mathscr{F}_k]_- = -\delta_{jk}\mathscr{F}_i$$
$$[[\mathscr{F}_i, \mathscr{F}_j]_-, \mathscr{F}_k]_- = 0, \qquad i, j, k = 1, 2, \ldots, n \tag{1.1}$$

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(which differ from the para-Fermi commutation relations only by the minus sign in the right-hand side of the first one) and the vacuum conditions

$$\mathscr{F}_i|0\rangle = 0, \qquad \mathscr{F}_i \overset{+}{\mathscr{F}}_j|0\rangle = p\delta_{ij}|0\rangle \tag{1.2}$$

$\mathscr{F}_i$, $\overset{+}{\mathscr{F}}_j$ are the annihilation and creation operators respectively, p is a positive number—the order of the related statistics. The particle number operator is defined as

$$N_i = \tfrac{1}{2}([\mathscr{F}_i, \overset{+}{\mathscr{F}}_i]_- - p) \tag{1.3}$$

It has been further shown (Kademova & Kraev, 1971c) that the algebra with $2n$ generators $\mathscr{F}_i$, $\overset{+}{\mathscr{F}}_j$ $(i, j = 1, 2, \ldots, n)$ defined by the commutation relations (1.1) is isomorphic to $O(2n, 1)$ algebra.

For the case $n = 1$ it has been proved (Kademova & Kraev, 1971a) that the representation space contains states with only positive norms for any $0 < p < \infty$. This result has been supposed to hold (Kademova & Kraev, 1971a, b, c) for $n > 1$ which, however, is not the case. Ohnuki *et al.* (1971) pointed out that the representation space of the commutation relations (1.1) combined with the vacuum conditions (1.2) for $n > 1$ contains some antisymmetric states with negative norms.

Thus the case of a single oscillator ($n = 1$) is singled out by the existence of Fock representations containing vectors with only positive norms, while for the case of a system of more than one oscillator ($n > 1$) Fock representations with only positive norms of the states do not exist.

Therefore if one uses the commutation relations (1.1) for quantising spin-half fields then one must use Fock representations with negative norms of some of the antisymmetric states or drop the vacuum conditions (1.2) and use essentially non-Fock representations—the infinite-dimensional unitary representations of $O(2n, 1)$.

One way of answering the question of the applicability of the quantisation scheme (1.1), without the vacuum conditions (1.2), is to see whether there are physical problems requiring essentially non-Fock representations of the commutation relations and whether the above-mentioned quantisation can be applied to such problems.

It has been shown (Schroer & Swieca, 1970; Schroer, 1971) that in the case of strong stationary external interactions of quantised fields and the quantisation of $m^2 < 0$ field equations the introduction either of negative metric or of the breakdown of the vacuum condition is essentially necessary.

Actually, all the quantum-mechanical problems which have $O(2, 1)$ as a spectrum-generating algebra (SGA) (Lánik, 1967, 1968, 1969, 1970; Cordero & Ghirardi, 1971; Cordero *et al.* 1971; Barut & Bornzin, 1971) can be treated in terms of a single oscillator, i.e. in terms of a single pair of creation and annihilation operators $\overset{+}{\mathscr{F}}$, $\mathscr{F}$ satisfying the commutation relations (1.1). The problems with small interactions can be solved within the Fock representations of the new commutation relations (for $n = 1$)

while in the case of strong attractive potentials one must use essentially non-Fock representations of these commutation relations (for $n = 1$).

2. $O(2,1)$ *Algebra Representations and Number Operators*

Here we shall briefly review some properties of $O(2,1)$ algebra representations and introduce corresponding particle number operators which will be used in the next section.

As is well known, $O(2,1) \sim SU(1,1)$ algebra is defined by the following commutation relations between its generators:

$$\mathscr{A}_i \qquad (i = 1, 2, 3)$$

$$[\mathscr{A}_1, \mathscr{A}_2]_- = -i\mathscr{A}_3, \qquad [\mathscr{A}_2, \mathscr{A}_3]_- = i\mathscr{A}_1, \qquad [\mathscr{A}_3, \mathscr{A}_1]_- = i\mathscr{A}_2 \quad (2.1)$$

It was shown (Kademova & Kraev, 1971c) that $\mathscr{A}_i$ can be expressed in terms of $\mathscr{F}$ and $\overset{+}{\mathscr{F}}$, satisfying (1.1) as follows:

$$\mathscr{A}_1 = \frac{\overset{+}{\mathscr{F}} + \mathscr{F}}{2}, \qquad \mathscr{A}_2 = \frac{\overset{+}{\mathscr{F}} - \mathscr{F}}{2i}, \qquad \mathscr{A}_3 = \frac{[\mathscr{F}, \overset{+}{\mathscr{F}}]_-}{2} \qquad (2.2)$$

The Casimir operator of the algebra is

$$\mathscr{T} = \mathscr{A}_3{}^2 - \mathscr{A}_1{}^2 - \mathscr{A}_2{}^2 \equiv \tfrac{1}{4}([\mathscr{F}, \overset{+}{\mathscr{F}}]_-{}^2 - 2[\mathscr{F}, \overset{+}{\mathscr{F}}]_+) \qquad (2.3)$$

The single-valued unitary irreducible representations (UIR) of the $SU(1,1)$ group were first constructed and studied by Bargmann (1947), who used a basis of the representation space in which the compact generator (the generator $\mathscr{A}_3$ of the compact $O(2)$ subalgebra) was diagonal. The same representations were studied further (Mukunda, 1967) in a basis in which a non-compact generator of $O(2,1) \sim SU(1,1)$ was diagonal.

The representations of the $SU(1,1)$ group in the standard Bargmann basis are split into the following classes depending on the values of the Casimir operator—$\mathscr{T} = j(j+1)$:

(A) The discrete class representations D_j^+ (D_j^-) are labelled by integer or half-integer values of the parameter $j = -\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots$, and the eigenvalues of the compact generator $\mathscr{A}_3$ are $m = -j, -j+1, \ldots$ ($m = j, j-1, \ldots$). For this class of representations the compact generator has a spectrum bounded below (above).

(B) The continuous class representations:

(a) The principal series C_j^δ

$$j = -\tfrac{1}{2} + is,$$
$$m = 0, \pm 1, \pm 2, \ldots, \qquad (\delta = 0) \qquad \text{for } 0 \leqslant s < \infty$$
$$m = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \ldots, \qquad (\delta = 1) \qquad \text{for } 0 < s < \infty$$

(b) The supplementary series

$$-\tfrac{1}{2} < j < 0, \qquad m = 0, \pm 1, \pm 2, \ldots$$

For these two series of representations the eigenspectrum of the compact generator is not bounded either below or above.

It is important for us that the $O(2,1) \sim SU(1,1)$ algebra has hermitian representations (which are, however, not true representations of the $SU(1,1)$ group) for all the values of $j < 0$ for which the spectrum of the compact generator is bounded below $m = -j, -j+1, \ldots$. These are exactly the Fock representations, labelled by $0 < p < \infty$, of the commutation relations (1,1) for $n = 1$ found in Kademova & Kraev (1971a). The eigenbasis of the compact generator $\mathscr{A}_3 = [\mathscr{F}, \overset{+}{\mathscr{F}}]_-/2$ is $(\overset{+}{\mathscr{F}})^n|0\rangle, n = 0, 1, 2, 3, \ldots$ The value of the Casimir operator for a representation labelled by a fixed p is $\mathscr{T} = (p/2)(p/2 - 1)$, i.e. $p/2 = -j$. (The true representations of the $SU(1,1)$ group are those labelled by integer p and can be realised in the Fock space of two Bose operators.)

(i) For this class of representations of the algebra $(0 < p < \infty)$ the operator

$$N_p = \mathscr{A}_3 + j = \tfrac{1}{2}([\mathscr{F}, \overset{+}{\mathscr{F}}]_- - p) \tag{2.4a}$$

has the eigenspectrum 0, 1, 2, ... and coincides with the particle number operator.

(ii) For the supplementary series and the principal series with $\delta = 0$ the compact generator

$$N \equiv \mathscr{A}_3 = \frac{[\mathscr{F}, \overset{+}{\mathscr{F}}]_-}{2} \tag{2.4b}$$

has the eigenspectrum $0, \pm 1, \pm 2, \ldots$.

(iii) For the principal series with $\delta = 1$ the operator

$$N = \mathscr{A}_3 + \tfrac{1}{2} = \tfrac{1}{2}([\mathscr{F}, \overset{+}{\mathscr{F}}]_- + 1) \tag{2.4c}$$

has the eigenspectrum $0, \pm 1, \pm 2, \ldots$, i.e. for the continuous class of representations the operator N has an eigenspectrum not bounded below. It consists of all integer numbers (negative as well as positive). This is the case of the particle number operator for the 'strange particle representations' (Chaiken, 1968).

The above representations were considered (Mukunda, 1967) also in the eigenbasis of the non-compact generator. The eigenspectrum of the non-compact generator $\mathscr{A}_1 = (\overset{+}{\mathscr{F}} + \mathscr{F})/2$ always consists of the entire real line. While for the representations of the discrete class $\mathscr{A}_1$ has a non-degenerate eigenspectrum, for the representations of the continuous class its eigenspectrum is twice degenerate.

3. *On the Connection between the Method of SGA and the New Quantisation*

The idea of using the SGA for solving different quantum-mechanical problems (Lánik, 1967, 1968, 1969, 1970; Cordero & Ghirardi, 1971;

Cordero *et al.*, 1971; Barut & Bornzin, 1971) consists in finding realisations of some Lie algebra in terms of operators of the Hilbert space of the system considered and expressing the Hamiltonian as a function of the generators of the algebra. The eigenspectrum of the Hamiltonian is found within a system of irreducible representations of the Lie algebra. It has been shown by the above authors that all quantum-mechanical problems that have been solved algebraically possess $O(2,1)$ as SGA. Due to the isomorphism of $O(2,1)$ with the algebra of the creation and annihilation operators $\overset{+}{\mathscr{F}}$, $\mathscr{F}$ of a single oscillator any of these quantum-mechanical problems can be treated as a single oscillator and the Hamiltonian H can be put in terms of the creation and annihilation operators

$$\mathscr{G}(H-\mathscr{E}) = \alpha[\mathscr{F}, \overset{+}{\mathscr{F}}]_- + \beta\overset{+}{\mathscr{F}} + \gamma\mathscr{F} + \lambda \tag{3.1}$$

(Following Cordero & Ghirardi (1971) $\mathscr{G}$ is an arbitrary non-singular operator; α, β, γ, λ are generally energy-dependent coefficients) where without loss of generality one can assume $\beta = \gamma$.

The particular values of α, β determine the possibility of transforming the right-hand side of equation (3.1) into either of the following forms:

$$\mathscr{G}(H-\mathscr{E}) \to \frac{[F, \overset{+}{F}]_-}{2} \tag{3.2a}$$

or

$$\mathscr{G}(H-\mathscr{E}) \to \frac{F+\overset{+}{F}}{2} \tag{3.2b}$$

(up to factors and additive constants).

As we know (Section 2) the operator $[\mathscr{F}, \overset{+}{\mathscr{F}}]_-/2$ has a discrete eigenspectrum and coincides, up to an additive constant, with the particle number operator. Thus (3.2a) gives the discrete spectrum of the system, while (3.2b) gives the continuous one, since $(\mathscr{F} + \overset{+}{\mathscr{F}})/2$ always has a continuous eigenspectrum.

Thus any quantum-mechanical problem with $O(2,1)$ as SGA can be considered as a single oscillator $(\overset{+}{\mathscr{F}}, \mathscr{F})$. Different quantum-mechanical problems require only different representations of the commutation relations (1,1) for a single oscillator ($n = 1$).

This can be illustrated by some simple examples:

(i) *Harmonic Oscillator*

The Hamiltonian of the harmonic oscillator in terms of Bose creation and annihilation operators $\overset{+}{a}$, a is

$$H = \hbar\omega(\overset{+}{a}a + \tfrac{1}{2}) \tag{3.3a}$$

Using the realisation (Kademova & Kraev, 1971a)

$$\mathscr{F} = \frac{a^2}{2}, \qquad \overset{+}{\mathscr{F}} = \frac{\overset{+}{a}{}^2}{2}$$

this Hamiltonian can be put in terms of $\mathscr{F}$ and $\overset{+}{\mathscr{F}}$ satisfying (1.1)

$$H = \hbar\omega[\mathscr{F}, \overset{+}{\mathscr{F}}]_- = \hbar\omega(2N_p + p) \tag{3.3b}$$

within the Fock representations $p = \frac{1}{2}$ and $\frac{3}{2}$ (Kademova & Kraev, 1971a).

(ii) *Non-relativistic Hydrogen Atom*

The problem is described by equation (3.1) with

$$\alpha = -\tfrac{1}{4} + 2\epsilon, \qquad \beta = \gamma = \tfrac{1}{4} + 2\epsilon, \qquad \lambda = 2z$$

and is solved within the following essentially Fock representations of a single oscillator (1.1) labelled by $p = 2, 4, 6, \ldots$.

For instance, the discrete spectrum which, in the usual notations, is given as

$$\mathscr{E} = \frac{-z^2}{2(n_r + l + 1)^2}, \qquad n_r = 0, 1, 2, \ldots, \qquad l = 0, 1, 2, \ldots$$

is expressed directly in terms of the eigenvalues of the particle number operator N_p within the above representations

$$\mathscr{E} = -\frac{z^2}{2\left(N_p' + \dfrac{p}{2}\right)}$$

where $p = 2(l + 1) = 2, 4, 6, \ldots$ and the spectrum of N_p' is $0, 1, 2, \ldots$.

All the other relativistic and non-relativistic problems (the Schrödinger equation for: the harmonic oscillator potential plus an extra cubic force, the hydrogen atom potential plus an extra cubic force etc.; the Klein–Gordon equation for the hydrogen atom; the second-order Dirac equation for the hydrogen atom, etc.) considered by the above authors can be solved within the Fock representations of the commutation relations (1.1) for $n = 1$.

It has been shown by Barut & Bornzin (1971) that the Klein–Gordon and the second-order Dirac equation can be solved within the discrete class representations of $O(2,1)$ algebra (i.e., they can be treated in terms of a single oscillator $\mathscr{F}$, $\overset{+}{\mathscr{F}}$ within the Fock representations) only in the case of a small coupling constant. When the coupling constant is very large these problems can be solved only by using the principal series of representations of $O(2,1)$ (Barut & Bornzin, 1971) (i.e., one must treat the problem in terms of a single oscillator $\mathscr{F}$, $\overset{+}{\mathscr{F}}$ using essentially non-Fock representations).

For instance, for the case of the Klein–Gordon equation with a strong attractive potential, the discrete eigenspectrum of the system is expressed in terms of the eigenspectrum of the operator $A_3 = [\mathscr{F}, \overset{+}{\mathscr{F}}]_-/2$, which coincides with the particle number operator up to an additive constant $p = \delta/2$ ($\delta = 0$ or 1), and is not bounded below within the 'strange particle representations',

$$\mathscr{E} = \frac{m}{\sqrt{1 + \left(\frac{\alpha}{A_3'}\right)^2}},$$

where α is the coupling constant and A_3' is the eigenspectrum of A_3.

Thus the quantum relativistic problems with very strong attractive potentials require essentially the use of non-Fock representations even for the lowest case of the new quantisation (1.1) (for $n = 1$).

This makes one hope that the non-Fock representations of the commutation relations (1.1) for $n > 1$ can have some physical applications.

4. *Para-Bose Realisation of the New Commutation Relations*

In Kademova & Kraev (1971a) explicit realisations of the commutation relations (1.1) for a single oscillator $(\mathscr{F}, \overset{+}{\mathscr{F}})$ were given as follows:

$$\mathscr{F} = \frac{a^2}{2}, \qquad \overset{+}{\mathscr{F}} = \frac{\overset{+}{a}{}^2}{2} \tag{4.1}$$

and

$$\mathscr{F} = ab, \qquad \overset{+}{\mathscr{F}} = \overset{+}{a}\overset{+}{b} \tag{4.2}$$

where a, $\overset{+}{a}$, b, $\overset{+}{b}$ are Bose operators.

Using (4.1) one realises the Fock representations of the commutation relations (1.1) labelled by $p = \frac{1}{2}$ and $\frac{3}{2}$ in the Fock space of a single Bose operator, and using (4.2) all the Fock representations with integer $p = 1, 2, 3, \ldots$ are realised in the Fock space of two Bose operators. (If a, $\overset{+}{a}$, b, $\overset{+}{b}$ in (4.1) and the symmetrized (4.2) are taken as para-Bose operators then $\mathscr{F}$ and $\overset{+}{\mathscr{F}}$ again satisfy the commutation relations (1.1).)

It is to be stressed here that the above realisations are valid only for a single oscillator. For more than one oscillator the operators $\mathscr{F}_i = a_i^2/2$ and $\overset{+}{\mathscr{F}}_j = \overset{+}{a}_j{}^2/2$, $i, j = 1, 2, \ldots, n$, (a_i, $\overset{+}{a}_j$ being para-Bose operators) satisfy not the commutation relations (1.1) but

$$\begin{aligned} &[\tfrac{1}{2}[\mathscr{F}_i, \overset{+}{\mathscr{F}}_i]_-, \mathscr{F}_i]_- = -\mathscr{F}_i \\ &\left.\begin{aligned}[\mathscr{F}_i, \mathscr{F}_j]_- = 0 \\ [\mathscr{F}_i, \overset{+}{\mathscr{F}}_j]_- = 0\end{aligned}\right\} i \neq j \end{aligned} \tag{4.3}$$

i.e., two different oscillators commute. The algebra determined by the commutation relations (4.3) is isomorphic to the direct product of n $O(2,1)$ algebras.

Here we would like to give an explicit realisation of a system of n oscillators satisfying the commutation relations (1.1).

In Kademova (1970), Kademova & Kálnay (1970), Kademova & Kraev (1970, 1971d) a general method for realising the para-Fermi algebra operators as polynomials of para-Bose and para-Fermi operators of arbitrary order of parastatistics has been worked out. It will be shown here that, using the finite-dimensional representations of the para-Fermi algebra with $2n$ generators, one can construct polynomials of para-Bose operators (Bose operators, in particular) which satisfy the new commutation relations (1.1).

Let a matrix representation of $2n$ para-Fermi operators be given, i.e. the 2^{np}-dimensional matrices F_α, $\overset{+}{F}_\beta$, $\alpha, \beta = 1, 2, \ldots, n$, satisfying the following commutation relations:

$$
\begin{aligned}
[\tfrac{1}{2}[F_\alpha, \overset{+}{F}_\beta]_-, F_\gamma]_- &= \delta_{\beta\gamma} F_\alpha \\
[[F_\alpha, F_\beta]_-, F_\gamma]_- &= 0
\end{aligned}
\tag{4.4}
$$

Using these matrices, we construct the following $2n$ polynomials of the para-Bose operators a_i, $\overset{+}{a}_j$, b_k, $\overset{+}{b}_l$, $i, j, k, l = 1, 2, \ldots, 2^{np}$, of arbitrary order of parastatistics q)†

$$
\begin{aligned}
\mathscr{F}_\alpha &= \tfrac{1}{2}(F_\alpha)_{ij}([\overset{+}{b}_i, \overset{+}{a}_j]_+ - [a_i, b_j]_+) \\
\overset{+}{\mathscr{F}}_\beta &= \tfrac{1}{2}(\overset{+}{F}_\beta)_{ij}([a_i, b_j]_+ - [\overset{+}{b}_i, \overset{+}{a}_j]_+)
\end{aligned}
\tag{4.5}
$$

Obviously

$$
(\mathscr{F}_\alpha)^+ = \overset{+}{\mathscr{F}}_\alpha
$$

It is easily shown that these polynomials satisfy the new commutation relations (1.1) and therefore are their realisation.

Indeed

$$
[\mathscr{F}_\alpha, \overset{+}{\mathscr{F}}_\beta]_- = \tfrac{1}{2}([F_\alpha, \overset{+}{F}_\beta]_-)_{ij}([\overset{+}{a}_j, a_i]_+ - [\overset{+}{b}_i, b_j]_+)
$$

and

$$
[\mathscr{F}_\alpha, \mathscr{F}_\beta]_- = -\tfrac{1}{2}([F_\alpha, F_\beta]_-)_{ij}([\overset{+}{a}_j, a_i]_+ - [\overset{+}{b}_i, b_j]_+)
$$

From here and from (4.4) we get

$$
\begin{aligned}
[\tfrac{1}{2}[\mathscr{F}_\alpha, \overset{+}{\mathscr{F}}_\beta]_-, \mathscr{F}_\gamma]_- &= -([\tfrac{1}{2}[F_\alpha, \overset{+}{F}_\beta]_-, F_\gamma]_-)_{ij}\tfrac{1}{2}([\overset{+}{b}_i, \overset{+}{a}_j]_+ - [a_i, b_j]_+) \\
&= -\delta_{\beta\gamma}\tfrac{1}{2}(F_\alpha)_{ij}([\overset{+}{b}_i, \overset{+}{a}_j]_+ - [a_i, b_j]_+) \\
&= -\delta_{\beta\gamma}\mathscr{F}_\alpha
\end{aligned}
$$

and

$$
[[\mathscr{F}_\alpha, \mathscr{F}_\beta]_-, \mathscr{F}_\gamma]_- = 0
$$

† We use the summation convention over the repeated indices.

Thus the above general procedure allows, knowing a particular representation of the para-Fermi algebra, classes of representations to be found of the new commutation relations (1.1). In other words, knowing some finite-dimensional hermitian representations of the $O(2n+1)$ algebra, one can construct some of the infinite-dimensional hermitian representations of the $O(2n, 1)$ algebra.

Such a particular realisation for the case $n = p = q = 1$ is

$$\mathscr{F} = \overset{+}{b}_1 \overset{+}{a}_2 - a_1 b_2, \qquad \overset{+}{\mathscr{F}} = b_1 a_2 - \overset{+}{a}_1 \overset{+}{b}_2 \tag{4.6}$$

For this case the realisations (4.1), (4.2),

$$\mathscr{F} = \tfrac{1}{2}(\overset{+}{a}_1{}^2 + \epsilon a_2{}^2), \qquad \overset{+}{\mathscr{F}} = \tfrac{1}{2}(a_1{}^2 + \epsilon \overset{+}{a}_2{}^2)$$

and

$$\mathscr{F} = \tfrac{1}{2}(a_1{}^2 + \epsilon a_2{}^2), \qquad \overset{+}{\mathscr{F}} = \tfrac{1}{2}(\overset{+}{a}_1{}^2 + \epsilon \overset{+}{a}_2{}^2), \qquad \epsilon = \pm$$

are simpler. However, the realisation (4.6) is important since it allows a simple extension to the case $n = 2$ as follows.

Let us use the matrix representation of two pairs of Fermi operators

$$F_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \overset{+}{F}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$F_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \overset{+}{F}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.7}$$

Then the operators defined through the mapping

$$\mathscr{F}_i = (a_1\, a_2\, \overset{+}{b}_1\, \overset{+}{b}_2)\, F_i \begin{pmatrix} \overset{+}{a}_1 \\ \overset{+}{a}_2 \\ -b_1 \\ -b_2 \end{pmatrix}, \qquad i = 1, 2 \tag{4.8}$$

$$\overset{+}{\mathscr{F}}_i = (\mathscr{F}_i)^+$$

($a_i, \overset{+}{a}_i, b_i, \overset{+}{b}_i$, $i = 1, 2$, are Bose operators), namely

$$\begin{aligned} \mathscr{F}_1 &= \overset{+}{b}_1 \overset{+}{a}_2 + a_1 b_2, \qquad & \overset{+}{\mathscr{F}}_1 &= b_1 a_2 + \overset{+}{a}_1 \overset{+}{b}_2 \\ \mathscr{F}_2 &= \overset{+}{b}_2 \overset{+}{a}_2 - a_1 b_1, \qquad & \overset{+}{\mathscr{F}}_2 &= b_2 a_2 - \overset{+}{a}_1 \overset{+}{b}_1 \end{aligned} \tag{4.9}$$

satisfy the commutation relations (1.1) for $n = 2$, i.e., they are their realisation in a space with a positive metric (the Fock space of four Bose operators). This is a realisation of some of the hermitian representations of $O(4,1)$ algebra in the Fock space of Bose operators.

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